

Operator Quantum Geometric Tensor and Quantum Phase Transitions

XIAO-MING LU and XIAOGUANG WANG ^(a)

Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou 310027, P.R. China.

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Abstract. - We extend the quantum geometric tensor from the state space to the operator level, and investigate its properties like the additivity for factorizable models and the splitting of two kinds contributions for the case of stationary reference states. This operator-quantum-geometric tensor (OQGT) is shown to reflect the sensitivity of unitary operations against perturbations of multi parameters. General results for the cases of time evolutions with given stationary reference states are obtained. By this approach, we get exact results for the rotated XY models, and show relations between the OQGT and quantum criticality.

Introduction. – The geometric structure on the manifold of quantum states (MQS) is an interesting subject of quantum physics, which was studied from different perspectives, e.g., the geometrization of quantum mechanics [1], the geometric phase [2–5], the geometry of quantum evolution [6], the quantum theory of gravity [7–9], quantum phase transitions [10, 11], etc. Generally speaking, on the MQS, there is a natural Riemann structure—the quantum-geometric tensor (QGT), induced by the inner product in the Hilbert space \mathcal{H} [5, 12]. The real part of the QGT provides the metric tensor, through which we can measure the distance between two states on the MQS. The imaginary part of QGT gives a curvature 2-form, corresponding to a natural symplectic structure on the projective Hilbert space $\mathcal{P}(\mathcal{H})$. With this curvature 2-form, the geometric phase can be interpreted as the area enclosed by the closed curve in $\mathcal{P}(\mathcal{H})$ [13].

The corresponding metric given by the QGT is a kind of Fubini-Study metric related to the fidelity of two quantum states, which is one of the most popular physical quantities in the quantum-information processing. Recently, it is proposed that the fidelity can characterize quantum phase transitions [14]. This approach has been achieved a lot of progress [10, 11, 14–30]. More recently, the fidelity approach was extended to the operator level—the operator fidelity (OF) [31, 32]. The underlying idea behind the OF is mapping the unitary operators into the quantum states.

It is proposed that the susceptibility (metric) of the OF can be used to study the stability of quantum evolution with respect to perturbations, indicate the quantum critical points, quantify the environment induced decoherence and investigate quantum chaos [31–35].

Since unitary operators can be mapped into quantum states, on the manifold of the unitary operators, we can define an operator-quantum-geometric tensor (OQGT), whose real and imaginary parts can give a metric tensor and a curvature 2-form respectively. Like the QGT is naturally induced by the inner product in the Hilbert space \mathcal{H} , the OQGT is naturally induced by the inner product of two operators, which is a generalization of the Hilbert-Schmidt product, with a reference state ρ [32]. Like the susceptibility of OF, The OQGT has the additivity for the factorizable model and the splitting of two kinds of contributions for the special cases that the reference state ρ is commutable with the operators. We will show this OQGT approach is useful to the sensitivity of unitary operations against multi-parameter perturbations.

Riemannian structure on the MQS. – The term MQS is pointed to the submanifold of the projective Hilbert space, which is defined as the sets of rays of the Hilbert space \mathcal{H} . The inner product structure of the Hilbert space \mathcal{H} naturally induce a Riemannian structure of the projective Hilbert space $\mathcal{P}(\mathcal{H})$ [12]. We denote the ray of \mathcal{H} by $|\tilde{\psi}\rangle$. The Fubini-Study metric indicating the

^(a)E-mail: xgwang@zimp.zju.edu.cn

distance between $|\tilde{\psi}\rangle$ and $|\tilde{\phi}\rangle$ is defined as follows

$$\gamma(\psi, \phi) = \arccos F, \quad (1)$$

where $F = |\langle\psi|\phi\rangle|$ is the fidelity defined through the inner product in \mathcal{H} . The infinitesimal form of this metric, which denote the distance between two slightly different states $|\tilde{\psi}(\lambda)\rangle$ and $|\tilde{\psi}(\lambda + d\lambda)\rangle$, can be written as

$$ds^2 \simeq 4(1 - F^2) = Q_{\mu\nu} d\lambda^\mu d\lambda^\nu, \quad (2)$$

where

$$Q_{\mu\nu} \equiv \langle \partial_\mu \psi | (1 - |\psi\rangle\langle\psi|) | \partial_\nu \psi \rangle \quad (3)$$

is the so-called quantum geometric tensor on $\mathcal{P}(\mathcal{H})$ [5, 12]. Here $\partial_\mu \equiv \partial/\partial\lambda^\mu$ with λ^μ -s the parameters characterizing the elements in $\mathcal{P}(\mathcal{H})$. Hereafter, we use the Einstein summation convention. $Q_{\mu\nu}$ is a Hermitian matrix, whose real part is symmetric while the imaginary part is anti-symmetric. Hence, only the real part of quantum geometric tensor has effects on Eq. (2) due to the summation, i.e., $ds^2 = g_{\mu\nu} d\lambda^\mu d\lambda^\nu$, where $g_{\mu\nu} = \text{Re } Q_{\mu\nu}$. Meanwhile, the imaginary part gives the curvature 2-form

$$\sigma = (\text{Im } Q_{\mu\nu}) d\lambda^\mu \wedge d\lambda^\nu, \quad (4)$$

where \wedge is the exterior (wedge) product. The integration of σ over an area S gives the geometric phase $\gamma_g = -\iint_S \sigma$ of the cycle evolution along the boundary of the area S [2, 4]. Further, σ can be written as the exterior differential of a 1-form—Berry's connection $\beta = -i\langle\psi|d\psi\rangle$, i.e., $\sigma = d\beta = -i\langle d\psi| \wedge |d\psi\rangle$.

Operator quantum geometric tensor. — In the space of operators, one can define the following Hermitian inner product

$$\langle X, Y \rangle_\rho := \text{Tr}(X^\dagger Y \rho), \quad (5)$$

where ρ is a density operator taken as the reference state. Especially, we concern the unitary operators hereafter, which satisfies the normalized condition $\langle U, U \rangle_\rho = 1$. Actually, this inner product structure can be interpreted as the conventional one for the two pure states $|X\rangle_\rho$ and $|Y\rangle_\rho$ in the extended Hilbert space [32], through a map from the operator X to the pure state $|X\rangle_\rho$ given as follows

$$X \mapsto |X\rangle_\rho \equiv X \otimes \mathbf{1} |\Psi(\rho)\rangle, \quad (6)$$

where $\mathbf{1}$ is the identity operator in the ancillary Hilbert space \mathcal{H}^{anc} and $|\Psi(\rho)\rangle$ is a pure state satisfying $\text{Tr}_{\text{anc}}(|\Psi(\rho)\rangle\langle\Psi(\rho)|) = \rho$, i.e., it is a purification of ρ in the extended Hilbert space. Obviously, the concrete form of purification does not influence the inner product (5). For brevity, we omit the subscript label ρ of $|X\rangle_\rho$ hereafter as long as not confusing.

Follow this spirit, the fidelity of two unitary operators is defined by

$$F(U_1, U_2) = |\text{Tr}(U_1^\dagger U_2 \rho)|, \quad (7)$$

which is equivalent to the fidelity $|\langle U_1 | U_2 \rangle|$ of the two pure states $|U_1\rangle$ and $|U_2\rangle$. If U_1 and U_2 are slightly different and ρ is pure, $F(U_1, U_2)$ is just the square root of the Loschmidt echo [36], which is used to describe the hypersensitivity of the time evolution to perturbations. It is remarkable that the minimization of $F(U_1, U_2)$ with respect to a optimal ρ is used to characterize the statistical distinguishability of two unitary operations U_1 and U_2 [37].

Like the Riemannian structure on the MQS [12], we can naturally induce the Riemann structure on the manifold of the unitary operators along with the given reference states. The main geometric object characterizes this structure is the quantum geometric tensor of the unitary operators

$$\begin{aligned} Q_{\mu\nu} &\equiv \langle \partial_\mu U | \partial_\nu U \rangle - \langle \partial_\mu U | U \rangle \langle U | \partial_\nu U \rangle \\ &= \langle A_\mu A_\nu \rangle_\rho - \langle A_\mu \rangle_\rho \langle A_\nu \rangle_\rho, \end{aligned} \quad (8)$$

where $A_\mu \equiv iU^\dagger \partial_\mu U$ is the Hermitian generator of displacements in λ^μ and $\langle O \rangle_\rho \equiv \text{Tr}[\rho O]$ denotes the expected value of the observable O on the state ρ . Hereafter, we call $Q_{\mu\nu}$ operator-quantum-geometric tensor (OQGT). The real part $g_{\mu\nu} = \text{Re } Q_{\mu\nu}$ is the Fubini-Study metric tensor of the unitary operator space with the reference state ρ , i.e., $ds^2 = g_{\mu\nu} d\lambda^\mu d\lambda^\nu$, which measure the statistical distance [38, 39] between the two states $U(\lambda) \otimes \mathbf{1} |\Psi(\rho)\rangle$ and $U(\lambda + d\lambda) \otimes \mathbf{1} |\Psi(\rho)\rangle$ in the extended Hilbert space. Meanwhile, due to the anti-commutative law of the exterior product, the imaginary part $\text{Im } Q_{\mu\nu} = \text{Im} \langle A_\mu A_\nu \rangle$ gives the curvature 2-form as follows

$$\sigma \equiv -i \langle A \wedge A \rangle_\rho, \quad (9)$$

where $A \equiv A_\mu d\lambda^\mu = iU^\dagger dU$ is a differential 1-form. If the reference states ρ are independent on λ^μ -s, σ can be written as $\sigma = d\beta$, where $\beta \equiv -\langle A \rangle_\rho$ is the analog of Berry's connection. Because β can be written as $\beta = -i\langle \Psi(\rho) | (U^\dagger \otimes \mathbf{I})(dU \otimes \mathbf{I}) | \Psi(\rho) \rangle$ with $|\Psi(\rho)\rangle$ the purification of ρ , it is, in fact, the Berry's connection in the extended Hilbert space. For a closed trajectory $\partial S = \{\lambda(t) : t \in [0, T]\}$ with $\lambda(0) = \lambda(T)$ in the parameter manifold, the geometric phase is given by $\gamma_g = -\oint_{\partial S} \beta = -\iint_S \sigma$, where S is the area subtended by the closed trajectory ∂S .

Next, we investigate the properties of the OQGT for the cases of factorizable models and of stationary reference states. These will be useful to the calculations in the XY model presented later.

Additivity for factorizable models. We analyze the case in which the unitary operator $U(\lambda)$ and the reference state $\rho(\lambda)$ have the same factorization structure as follows

$$U(\lambda) = \bigotimes_{k=0}^M U_k(\lambda), \quad \rho(\lambda) = \bigotimes_{k=0}^M \rho_k(\lambda). \quad (10)$$

This assumption is sensible when $U(\lambda)$ and $\rho(\lambda)$ are both generated by the same Hamiltonian. For instance, $U = \exp(-itH(\lambda))$ is the time evolution generated by $H(\lambda)$

and $\rho(\lambda)$ is the Gibbs thermal state which is a mixture of the eigenstates of $H(\lambda)$. After derivative with respect to λ^μ we obtain

$$\partial_\mu U(\lambda) = \sum_{l=0}^M [\bigotimes_{k=0}^{l-1} U_k(\lambda)] \otimes \partial_\mu U_l(\lambda) \otimes [\bigotimes_{k=l+1}^M U_k(\lambda)]. \quad (11)$$

Because $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$ and $\text{Tr}(\rho_l) = 1$, we get

$$\begin{aligned} \langle \partial_\mu U | \partial_\nu U \rangle &= \sum_l \langle \partial_\mu U_l | \partial_\nu U_l \rangle_l + \sum_{l \neq l'} M_{ll'}^{\mu\nu}, \\ \langle \partial_\mu U | U \rangle \langle U | \partial_\nu U \rangle &= \sum_l \langle \partial_\mu U_l | U_l \rangle_l \langle U_l | \partial_\nu U_l \rangle_l + \sum_{l \neq l'} M_{ll'}^{\mu\nu}, \end{aligned}$$

where $M_{ll'}^{\mu\nu} = \text{Tr}_l[\rho_l(\partial_\mu U_l)^\dagger U_l] \text{Tr}_{l'}[\rho_{l'} U_{l'}^\dagger(\partial_\nu U_{l'})]$, and $\langle A | B \rangle_l = \text{Tr}_l[\rho_l A^\dagger B]$, the subscript l means the l -th subspace. In such cases, we have

$$Q_{\mu\nu} = \sum_l Q_{\mu\nu}^l, \quad (12)$$

with $Q_{\mu\nu}^l = \langle \partial_\mu U_l | \partial_\nu U_l \rangle_l - \langle \partial_\mu U_l | U_l \rangle_l \langle U_l | \partial_\nu U_l \rangle_l$. Hence, for the model whose unitary operators and the reference state ρ have the same factorization structure, the OQGT is additive.

Splitting for stationary reference states. For matrix which can be diagonalized, the variance with respect to the parameters can be decomposed into two parts, the variance of the eigenvalues and of eigenvectors. A unitary operator $U(\lambda)$ can always be written in the form $U(\lambda) = S^\dagger(\lambda) U_d(\lambda) S(\lambda)$, where $U_d(\lambda)$ is a diagonal unitary matrix and $S(\lambda)$ is unitary. So the derivative of U can be splitted into two terms as follows

$$\partial_\mu U = D_\mu^{(1)} + D_\mu^{(2)} \quad (13)$$

with $D_\mu^{(1)} \equiv S^\dagger(\partial_\mu U_d) S$ and $D_\mu^{(2)} \equiv -i[U, \mathbf{a}_\mu]$, where $\mathbf{a}_\mu \equiv iS^\dagger \partial_\mu S$ is a Hermitian matrix. Note that we have $[D_\mu^{(1)}, U] = 0$ and $[D_\mu^{(1)\dagger}, U] = 0$, since $D_\mu^{(1)}$ and U can be simultaneously diagonalized by the unitary operator $S(\lambda)$. Correspondingly, we have $A_\mu = A_\mu^{(1)} + A_\mu^{(2)}$ with $A_\mu^{(i)} = iU^\dagger D_\mu^{(i)}$ for $i = 1, 2$.

For cases in which the reference states $\rho(\lambda)$ is stationary under the unitary operation $U(\lambda)$, i.e., $[\rho, U] = 0$, we chose $S(\lambda)$ as the unitary matrix diagonalizing simultaneously $U(\lambda)$ and $\rho(\lambda)$. Then we have

$$\begin{aligned} \langle A_\mu^{(2)} \rangle_\rho &= \text{Tr} \{ U^\dagger [U, \mathbf{a}_\mu] \rho \} = 0, \\ \langle A_\mu^{(1)\dagger} A_\nu^{(2)} \rangle_\rho &= -i \text{Tr} \{ D_\mu^{(1)\dagger} [U, \mathbf{a}_\nu] \rho \} = 0. \end{aligned} \quad (14)$$

The second equality of second line of the above equations is due to the commutation relation $[D_\mu^{(1)\dagger}, U] = 0$ and $[\rho, U] = 0$. Then substituting $A_\mu = A_\mu^{(1)} + A_\mu^{(2)}$ into Eq. (8) and combining Eq. (14), we obtain the splitted form of the OQGT as follows

$$\begin{aligned} Q_{\mu\nu} &= Q_{\mu\nu}^{(1)} + Q_{\mu\nu}^{(2)}, \\ Q_{\mu\nu}^{(1)} &= \langle A_\mu^{(1)} A_\nu^{(1)} \rangle_\rho - \langle A_\mu^{(1)} \rangle_\rho \langle A_\nu^{(1)} \rangle_\rho, \\ Q_{\mu\nu}^{(2)} &= \langle A_\mu^{(2)} A_\nu^{(2)} \rangle_\rho. \end{aligned} \quad (15)$$

Note that $[A_\mu^{(1)}, A_\nu^{(1)}] = 0$ and the Hermitian of $A_\mu^{(1)}$, so $Q_{\mu\nu}^{(1)}$ is real. Then, the curvature 2-form can be expressed as

$$\sigma = -i \langle A^{(2)} \wedge A^{(2)} \rangle_\rho \quad (16)$$

with $A^{(i)} \equiv A_\mu^{(i)} d\lambda^\mu$.

A motivation of such separation may be seen from time dependence of $Q^{(1)}$ and $Q^{(2)}$ if we consider the time evolution operator $U(\lambda) = \exp[-itH(\lambda)]$ [32]. $\hbar = 1$ is assumed hereafter. We assume the Hamiltonian $H(\lambda)$ and the reference state $\rho(\lambda)$ can be simultaneously diagonalized by unitary matrix $S(\lambda)$, i.e., $H_d(\lambda) = S(\lambda)H(\lambda)S^\dagger(\lambda)$ and $\rho_d = S(\lambda)\rho(\lambda)S^\dagger(\lambda)$ are both of diagonal form. Correspondingly, we have $U(\lambda) = S^\dagger(\lambda)U_d(\lambda)S(\lambda)$ with $U_d(\lambda) = \exp[-itH_d(\lambda)]$. Substituting the time evolution form of $U(\lambda)$ into Eq. (15), we obtain

$$\begin{aligned} Q_{\mu\nu}^{(1)} &= \alpha_{\mu\nu} t^2, \\ Q_{\mu\nu}^{(2)} &= \sum_{ij} \beta_{\mu\nu}^{ij} \{1 - \cos[(E_i - E_j)t]\}, \end{aligned} \quad (17)$$

where E_i is the eigenvalue of H . $\alpha_{\mu\nu}$, $\beta_{\mu\nu}^{ij}$ are both time-independent and defined by

$$\begin{aligned} \alpha_{\mu\nu} &\equiv \text{Tr}[(\partial_\mu H_d)(\partial_\nu H_d)\rho_d] \\ &\quad - \text{Tr}[(\partial_\mu H_d)\rho_d] \text{Tr}[(\partial_\nu H_d)\rho_d], \\ \beta_{\mu\nu}^{ij} &\equiv 2\rho_i[\mathbf{a}_\mu]_{ij}[\mathbf{a}_\nu]_{ji}, \end{aligned} \quad (18)$$

where $[\mathbf{a}_\mu]_{ij}$ is the matrix element of \mathbf{a}_μ in the representation whose basis are the eigenvectors of H . So for such cases, $Q_{\mu\nu}^{(1)}$ is proportional to t^2 , while $Q_{\mu\nu}^{(2)}$ consists of circular functions. It is remarkable that for pure states, $\alpha_{\mu\nu}$ is vanished, only the second term of Eq. (15) exists.

XY model. – The one-dimension spin-1/2 XY model, with $N = 2M + 1$ spins, in the presence of a transverse magnetic field characterized by λ , is described by the Hamiltonian

$$H_{\text{XY}} = - \sum_{l=-M}^M \left[\frac{1+\gamma}{2} \sigma_l^x \sigma_{l+1}^x + \frac{1-\gamma}{2} \sigma_l^y \sigma_{l+1}^y + \lambda \sigma_l^z \right], \quad (19)$$

where σ_l^a ($a = x, y, z$) are the Pauli matrix for the l -th spin and γ represents the anisotropy in the $x - y$ plane. The periodic condition ($M + 1 = -M$) is assumed here. We are interested in a new family of Hamiltonians obtained by applying a rotation of ϕ around the z direction to each spin

$$H(\phi, \gamma, \lambda) = g^\dagger(\phi) H_{\text{XY}}(\gamma, \lambda) g(\phi), \quad (20)$$

where $g(\phi) = \prod_{l=-M}^M \exp(i\phi \sigma_{lz}/2)$. This model was used to establish a relation between geometric phases and criticality of spin chains [40–42].

To obtain the analytic solution of this model, we apply three steps of transformations as follows: (i) the Jordan-Wigner transformation [43] $\sigma_i^+ = \prod_{j<i} (1 - 2c_j^\dagger c_j) c_i$, $\sigma_i^z = 1 - 2c_i^\dagger c_i$; (ii) Fourier transformation

$c_l = (1/\sqrt{N}) \sum_k e^{i2\pi kl/N} d_k$ with $k = -M, \dots, M$; (iii) the pseudo Pauli operator transformation defined by [44]

$$\begin{aligned}\sigma_{kx} &= d_k^\dagger d_{-k}^\dagger + d_{-k} d_k, \\ \sigma_{ky} &= -id_k^\dagger d_{-k}^\dagger + id_{-k} d_k, \\ \sigma_{kz} &= d_k^\dagger d_k + d_{-k}^\dagger d_{-k} - 1,\end{aligned}\quad (21)$$

which satisfy the commutative relations $\{\sigma_{ka}, \sigma_{kb}\} = 2\delta_{ab}$ and $[\sigma_{ka}, \sigma_{kb}] = 2i\varepsilon_{abc}\sigma_{kc}$, where ε_{abc} is the Levi-Civita symbol. For $k \neq 0$, each of them acts on the Fock space spanned by $\{|\text{vac}\rangle_{k,-k}, |k, -k\rangle, |k\rangle, |-k\rangle\}$ with $|\text{vac}\rangle$ the vacuum state for d_k and d_{-k} fermions. These pseudo Pauli operators become conventional Pauli matrix in the sub-space spanned by $\{|0\rangle_{k,-k}, |k, -k\rangle\}$ and vanish elsewhere.

After these three steps, we get

$$\begin{aligned}H &= (\lambda - 1)\sigma_{0z} + \sum_{k=-M}^M H_{kz}, \\ H_{kz} &= S_k^\dagger(\phi, \theta_k) H_{k,d}(\Lambda_k) S_k(\phi, \theta_k),\end{aligned}\quad (22)$$

where $H_{k,d} = \Lambda_k \sigma_{kz}$ and $S_k(\phi, \theta_k) = R_{kx}(\theta_k) R_{kz}(\phi)$ with $R_{ka}(\alpha) = \exp(-i\alpha\sigma_{ka}/2)$ for $a = \{x, y, z\}$. The intermediate parameters Λ_k and θ_k are given by

$$\begin{aligned}\Lambda_k &= 2\sqrt{(\lambda - \cos(2\pi k/N))^2 + \gamma^2 \sin^2(2\pi k/N)}, \\ \theta_k &= -\frac{i}{2} \ln \frac{\lambda - \cos(2\pi k/N) + i\gamma \sin(2\pi k/N)}{\lambda - \cos(2\pi k/N) - i\gamma \sin(2\pi k/N)}.\end{aligned}\quad (23)$$

We consider the OQGT of the time evolution operators $U = \exp(-itH)$ and chose the ground state $\rho = |G\rangle\langle G|$ as the reference state. They are of the same factorization structure

$$\begin{aligned}U &= \bigotimes_{k=0}^M S_k^\dagger(\phi, \theta_k) \exp(-itH_{k,d}) S_k(\phi, \theta_k), \\ \rho &= \bigotimes_{k=0}^M S_k^\dagger(\theta_k) |\downarrow\rangle_k \langle\downarrow| S_k(\theta_k),\end{aligned}\quad (24)$$

where $|\downarrow\rangle_k$ is the eigenstate of σ_{kz} with eigenvalue -1 . Due to this factorization structure, the OQGT is additive. Because $[\rho, U] = 0$ and ρ is pure state, $Q_{\mu\nu}$ is splitted and $Q_{\mu\nu}^{(1)}$ of Eq. (17) vanishes. To obtain the $Q_{\mu\nu}^{(2)}$, we first get the 1-form matrix $\mathbf{a} \equiv iS^\dagger dS$, which is given by $\mathbf{a} = \sum_{k=0}^M \mathbf{a}_k$ with

$$\begin{aligned}\mathbf{a}_k &= iS_k^\dagger(\phi, \theta_k) d[S_k(\phi, \theta_k)] \\ &= \frac{d\theta_k}{2} R_{kz}^\dagger(\phi) \sigma_{kx} R_{kz}(\phi) + \frac{d\phi}{2} \sigma_{kz}.\end{aligned}\quad (25)$$

Substituting Eq. (25) into Eq. (17), we obtain the compo-

nents of the OQGT as follows

$$\begin{aligned}Q_{\lambda\lambda} &= \sum_{k=1}^M \frac{4}{\Lambda_k^2} \sin^2(\Lambda_k t) \sin^2 \theta_k, \\ Q_{\gamma\gamma} &= \sum_{k=1}^M \frac{4}{\Lambda_k^2} \sin^2(\Lambda_k t) \cos^2 \theta_k \sin^2\left(\frac{2\pi k}{N}\right), \\ Q_{\phi\phi} &= \sum_{k=1}^M \sin^2(\Lambda_k t) [\sin(2\theta_k)]^2, \\ Q_{\lambda\gamma} &= \sum_{k=1}^M -\frac{4}{\Lambda_k^2} \sin^2(\Lambda_k t) \sin \theta_k \cos \theta_k \sin\left(\frac{2\pi k}{N}\right), \\ Q_{\lambda\phi} &= \sum_{k=1}^M -i \frac{2}{\Lambda_k} \sin^2(\Lambda_k t) \sin \theta_k \sin(2\theta_k), \\ Q_{\gamma\phi} &= \sum_{k=1}^M i \frac{2}{\Lambda_k} \sin^2(\Lambda_k t) \cos \theta_k \sin(2\theta_k) \sin\left(\frac{2\pi k}{N}\right).\end{aligned}$$

With the help of the metric tensor $g_{\mu\nu} \equiv \text{Re } Q_{\mu\nu}$, we can obtain the Loschmidt echo $L = |\langle G(x) | U^\dagger(x + \delta x) U(x) | G(x) \rangle|^2$ for arbitrary multi-parameter perturbations $\delta x = (\dots, \delta x^\mu, \dots)$. This can be seen from Eq. (2), then we have $L = 1 - g_{\mu\nu} \delta x^\mu \delta x^\nu / 4$. So the metric tensor directly reflects the sensitivity of the time evolution to the perturbations of multi parameters. A relation between the Loschmidt echo and quantum criticality was established in Ref. [45].

The critical points of the XY model are given by the lines $\lambda = \pm 1$ and by the segment $|\lambda| < 1, \gamma = 0$ [46]. We consider the region of the Ising-type phase transition at $\lambda = 1$. The time evolutions of the components of the metric tensor $g_{\mu\nu} = \text{Re } Q_{\mu\nu}$ for different values of λ are shown in Fig. (1), at $\gamma = 1$ for simplicity. The behaviors of the λ -related components $g_{\lambda\lambda}$ and $g_{\lambda\gamma}$ are obviously singular at the critical point $\lambda = 1$. $g_{\lambda\lambda}$ abruptly increases with time near the critical point, which implies the time evolution is highly sensitive in this critical point. Near the critical point, $g_{\lambda\gamma}$ also has a qualitative distinction, it sharply decreases to negative values with time when λ slightly greater than 1.

In the three-dimensional manifold of the parameter (λ, γ, ϕ) , the curvature 2-form σ is equivalent to a effective vector field $\mathbf{B} = (B_\lambda, B_\gamma, B_\phi)$ through the relation

$$\sigma = B_\lambda d\lambda^\gamma \wedge d\lambda^\phi + B_\gamma d\lambda^\phi \wedge d\lambda^\lambda + B_\phi d\lambda^\lambda \wedge d\lambda^\gamma. \quad (26)$$

In this rotated XY model, we obtain $\mathbf{B} = (2\text{Im}Q_{\gamma\phi}, -2\text{Im}Q_{\lambda\phi}, 0)$. For the case of $\gamma = 1$, the time evolution of B_γ ($-2\sigma_{\lambda\phi}$) is very different between the two sides of the critical point, as shown in Fig. (2). When λ is slightly less than 1, B_γ sharply increases with time, meanwhile when λ is slightly greater than 1, B_γ sharply decreases from zero to negative values. So we conclude that the amplitude of B_γ sharply increases near the critical but the direction of B_γ is opposite in the different phases.

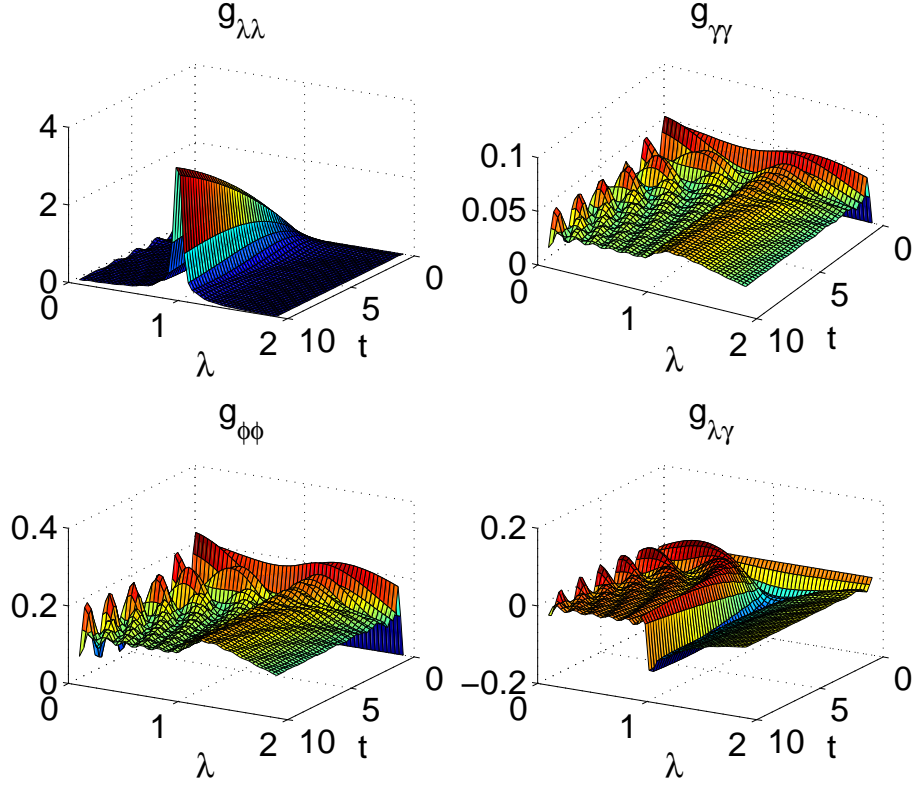


Fig. 1: Three dimensional diagram of the components of the rescaled metric tensor $g_{\mu\nu} = \text{Re } Q_{\mu\nu}/N$ as functions of λ and t for the system at $\gamma = 1$ with $N = 1001$ the number of spins. Here μ, ν are chosen from the parameter set $\{\lambda, \gamma, \phi\}$.

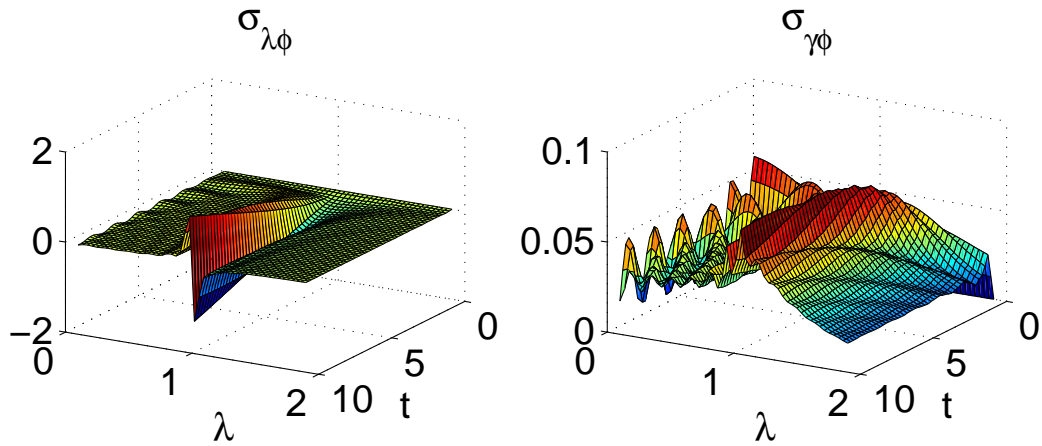


Fig. 2: Three dimensional diagram of the components of the rescaled curvature tensor $\sigma_{\mu\nu} = \text{Im } Q_{\mu\nu}/N$ as functions of λ and t for the system at $\gamma = 1$ with $N = 1001$ the number of spins. Here μ, ν are chosen from the parameter set $\{\lambda, \gamma, \phi\}$.

Conclusion. — To summarize, we have introduced the operator-quantum-geometric tensor on the manifold of the unitary operators with a given reference state, and prove the additivity for factorizable models. We show the splitting of two kinds contributions to the OQGT for the cases of stationary reference states, and these two contributions have different type of time dependence. This OQGT approach can be applied to investigate the sensitivity of unitary operation against perturbations. The Loschmidt echo for the unperturbed and perturbed time evolutions for arbitrary kind of multi-parameter perturbations can be easily got as long as the OQGT is obtained. We used this approach to investigate sensitivity of the time evolutions of the rotated XY models and show the sharp changes of the OQGT near the quantum critical points. We believe that this OQGT approach is useful to some interesting questions, like the quantum criticality [31, 33], decoherence [32] and quantum chaos [34, 35], etc. After the acceptance of this paper, we notice the work [47] where both the QGT and Operator fidelity are shown to be relevant to the adiabatic error in the adiabatic and holonomic quantum computing.

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